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Essential Form of General Uncertainty Distributions

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Abstract: In current uncertainty modeling literature, people almost got used to engage probability theory and statistics to handle uncertainty modeling problems. This paper addresses the necessity of engaging uncertainty distributions in modeling. In probability, distribution theory plays a central role in its developments and applications facilitated by mathematical statistics. A natural way in developing uncertainty theory is what an uncertainty distribution should be and the role it should play in the new subject – Uncertainty Statistics. In this paper, we review the axioms of $\sigma$-sub-additive uncertain measure proposed by Professor Baoding Liu in 2009, concept of uncertainty variable and uncertainty distribution, and then investigate the necessary and sufficient conditions and the formality for a function to define an uncertainty variable in a comparative manner to probability distribution.

Keywords: probability distribution, random variable, uncertain distribution, uncertain measure, uncertain variable, identification function

I. Introduction

Liu [6, 7] recently proposed an axiomatic uncertain measure theory and established quickly an uncertain theoretic system, including uncertain processing, uncertain processes, uncertain logic, uncertain inference, etc. for modeling uncertain phenomena in the real world.

It is well-known that probability theory is well-accepted and the most often used tool for analyzing, explaining, and modeling real-world uncertain (random) phenomena successfully.

However, the probability measure upon which a random variable is characterized and axiomatically founded with a key characteristic - $\sigma$-additivity. In dealing real world modeling, it is often discovered that $\sigma$-additivity is too restrictive to implement it. Furthermore, it was gradually discovered that randomness is not the only species of uncertainty so that other forms of uncertainty, say, vagueness, roughness, greyness and etc got more and more attention. As Zadeh [5] pointed out, "it has become increasingly clear that there are some important facets of uncertainty which do not lend themselves to analysis by classical probability-based methods".

Fuzziness, which is logically the break down of the law of excluding the middle, was first defined by Zadeh [5] in membership formality and the fuzzy mathematics was quickly developed and applied in the engineering and management. With many fuzzy measure theoretic developments, Liu [4, 5] proposed and established axiomatic credibility measure theory and even further developed chance measure theory for modeling hybrid uncertain events in real world.

The classical measure and probability measure belong to the class of “completely additive measure”, i.e., the measure of union of disjoint events is just the sum of the measures. In contrast, capacity, belief measure, plausibility measure, fuzzy measure, possibility measure and necessity measure belong to the class of “completely non-additive measures”. Since this class of measures does not assume the self-duality property, all of those measures are inconsistent with the law of contradiction and law of excluded middle, which dominate human thinking logic.

Uncertain measure is neither a completely additive measure nor a completely non-additive measure. In fact, uncertain measure is a “partially additive measure” because of its self-duality. Credibility measure and chance measure are special types of uncertain measure. One advantage of this class of measures is the sound consistency with the law of contradiction and law of excluded middle.

It is natural question how the uncertain phenomena are characterized mathematically and what is the essential form of an uncertain distribution which reflects the uncertain measure with $\sigma$-subadditivity property underlying the distribution?

The answer is fairly straightforward for the first part of the question – an uncertain distribution is used for characterization because an distribution fully characterize the uncertain law by which the uncertain variable takes it values and it is a familiar (point) function on the Euclidean space which is better mastered than the set function on the abstract space. But the second part is complicated.

In elementary or medium (even advanced) applied probability and statistics courses, distribution functions “dominate” the developments and applications. Just as Ash [1] pointed out, “Very often, the following statement is made: ‘Let $X$ be a random variable with distribution function $F'$, where $F$ is a given function from $\mathbb{R}$ to $[0, 1]$ that is increasing and right continuous, with $F(\infty) = 1, F(-\infty) = 0$.

There is no reference to the underlying probability space \((\Omega, \mathcal{F}, P)\), actually the \(F\) determines the probability measure \(P\), which in turn determines the probability of all events involving \(X\).

Nevertheless, the above practices do not mean the measure defined on abstract set class (space) can be ignored. Measure defines an event measuring grade system for extracting a conceptual uncertain environment.

In this paper, we investigate the essential form of an uncertain distribution with a \(\sigma\)-subadditivity uncertain measure underlying it by reviewing and comparing probability distribution and uncertain distribution systematically.

The structure of the remaining sections is stated as follows. Section II will be used to review Kolmogorov’s [3] probability measure axioms and measurable mapping of random variable and the associated probability distribution. Also the converse problem is investigated: given a function (satisfying the necessary and sufficient conditions), how a Lebesgue-Stieltjes measure is defined for characterizing a random variable following Ash [1]. In Section III, we reviewed Liu’s [4] uncertain measure axioms and concept of uncertain measurable mapping, uncertain variable and associated uncertain distribution. Section IV serves the discussion on the necessary and sufficient conditions for a function to be qualified as an uncertain distribution based on Peng and Iwamura’s [10]. Furthermore, we give a general definition of an uncertain distribution. In Section V, the formality of uncertain distribution is discussed in terms of Liu’s [6] identification functions. In Section VI, we further discuss the essential form of an uncertain distribution. Section VII concludes the paper.

II. Probability Distribution

A fundamental question is what kind of function can define a probability measure and thus define a random variable? To address this question, we need to start with Kolmogorov’s [3, 11] Axioms of probability.

Let \(\Omega\) be a nonempty set (space), and \(\mathcal{F}(\Omega)\) the \(\sigma\)-algebra on \(\Omega\). Each element, let us say, \(A \subset \Omega, A \in \mathcal{F}(\Omega)\) is called an uncertain event. A number denoted as \(P\{A\}\), \(0 \leq P\{A\} \leq 1\), is assigned to event \(A \in \mathcal{F}(\Omega)\), which indicates the uncertain measuring grade with which event \(A \in \mathcal{F}(\Omega)\) occurs. The normal set function \(P\{A\}\) satisfies following axioms given by Kolmogorov [3]:

**Axiom K.1**: (Nonnegativity) The probability of an event is a nonnegative real number, i.e., \(P\{A\} \geq 0\), \(\forall A \in \mathcal{F}\).

**Axiom K.2**: (Unit measure) The probability of the entire sample space is 1, i.e., \(P\{\Omega\} = 1\)

**Axiom K.3**: (\(\sigma\)-Additivity) Any countable sequence of pairwise disjoint events \(A_i, A_j, \cdots \in \mathcal{F}\) \(A_i \cap A_j = \emptyset, (i \neq j, 1, 2, \cdots)\) satisfies,

\[
P\left( \bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} P\{A_n\} \]

The direct consequences of the three axioms are stated as following:

**Theorem 2.1**: (Additive law of probability) Let \(A_i, A_j \in \mathcal{F}\), then

\[
P\{A_i \cup A_j\} = P\{A_i\} + P\{A_j\} - P\{A_i \cap A_j\} \]

**Theorem 2.2**: (Self-Duality) For \(\forall A \in \mathcal{F}\),

\[
P\{A^c\} = 1 - P\{A\} \]

**Definition 2.3**: (Primas [11]) Any set function \(P: \mathcal{F} \rightarrow [0,1]\) satisfies Axioms K.1-K.3 is called a probability measure. The triple \((\Omega, \mathcal{F}, P)\) is called the uncertain measure space.

**Definition 2.4**: An random variable \(X\) is a measurable mapping, i.e., \(X: (\Omega, \mathcal{F}(\Omega)) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))\), where \(\mathcal{B}(\mathbb{R})\) denotes the Borel \(\sigma\)-algebra on \(\mathbb{R} = (\infty, +\infty)\). To understand the measurability of a random variable, particularly, the role played by the \(\sigma\)-algebra \(\mathcal{F}(\Omega)\), let us recall however the measurability structured for a random variable. Let \((\Omega, \mathcal{F}, P)\) be a probability space and \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\) be a measurable space on real-line, then a real-valued function \(X\) is random variable if and only if for all \(r \in \mathbb{R}\), the pre-image \(\{\omega \in \Omega: X(\omega) \leq r\} \in \mathcal{F}\). For each value \(r \in \mathbb{R}\) taken by a real-valued random variable \(X\), the event \(B = (\infty, r]\) is an element of the Borel \(\sigma\)-algebra of real-line \(\mathbb{R}\), the pre-image of event \(B\) under random variable \(X\) is

\[
\{\omega \in \Omega: X(\omega) \in B\} = \{\omega \in \Omega: X(\omega) \leq r\} \]

(4)

event \(\{\omega \in \Omega: X(\omega) \leq r\}\) is an element of \(\sigma\)-algebra \(\mathcal{F}\) of \(\Omega\), where the probability measure \(P\) defined on this set class, i.e., \(\sigma\)-algebra \(\mathcal{F}\), i.e., \(P: \mathcal{F} \rightarrow [0,1]\).

Therefore every element (event) of \(\mathcal{F}\) is assigned with a probability grade, i.e., event \(\{\omega \in \Omega: X(\omega) \leq r\}\) is assigned a probability grade, which is \(P\{\omega \in \Omega: X(\omega) \leq r\}\).
Overall, $\sigma$-algebra $\mathcal{F}$ facilitates the formal definition of a random variable in terms of membership of the pre-image $\{\omega \in \Omega : X(\omega) \leq r\}$ to the $\sigma$-algebra $\mathcal{F}$, in which the probability measuring grade defined and every event of $\sigma$-algebra $\mathcal{F}$ is assigned. As Chung [2] pointed, each random variable on the probability space $(\Omega, \mathcal{F}, P)$ induces a probability space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ by means of the following correspondence.

$$\forall B \in \mathcal{B}(\mathbb{R}): \mu(B) = P\{X^{-1}(B)\} = P\{X \in B\} \quad (5)$$

Chung [2] further denotes $\mu = P \circ X^{-1}$ and specifically, the probability distribution is defined by the induced measure $\mu$,

$$F(r) = \mu\{(\infty, r]\} = P\{X \leq r\} \quad (6)$$

In conclusion, the random variable $X$ defined on a given probability space $(\Omega, \mathcal{F}(\Omega), P)$ is a measurable mapping to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and thus induces the distribution function, $F$, which is used to characterize the random variable.

Conversely, a nonnegative real-valued function $F$, given the function satisfying some conditions, a measure can be defined by $F$, and accordingly a random variable $X$ on an appropriate probability space.

**Definition 2.5**: The function, denoted by $F$ is a probability distribution function if and only $F$ satisfies following three conditions:

1. $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to +\infty} F(x) = 1$;
2. $F(x)$ is non-decreasing in $x$;
3. $F$ is right-continuous, i.e., $\forall x_0 \in \mathbb{R}$, $\lim_{x \downarrow x_0} F(x) = F(x_0)$.

**Definition 2.6**: A function $\Psi$ is normed if and only if it is mapped from real-line to unit interval $[0, 1]$, i.e.,

$$\Psi: \mathbb{R} \to [0, 1] \quad (16)$$

**Remark 2.7**: It is obvious that distribution is a right-continuous non-decreasing normed function defined on $\mathbb{R}$.

**Definition 2.8**: (Ash [1]) A Lebesgue-Stieltjes measure is a set function $\mu$ on Borel $\sigma$-algebra $\mathcal{B}(\mathbb{R})$ such that $\mu\{I\} = +\infty$ for each bounded interval $I \subset \mathbb{R}$.

**Definition 2.9**: (Ash [1]) Let $\mathcal{F}$ be a set class of a space (set) $\mathcal{G}$. Then $\mathcal{G}$ is termed as an algebra (field) if and only if $\mathcal{S} \in \mathcal{G}$ and $\mathcal{G}$ is closed under complementation and finite union.

**Theorem 2.10**: (Ash [1]) Let $F$ be a right-continuous non-decreasing normed function on the compact space $\mathbb{R}$, define

$$\mu\{(a, b]\} = F(b) - F(a), \forall a, b \in \mathbb{R} \quad (7)$$

Further, define

$$\mu\{\bigcup_{k=1}^{n} I_k\} = \sum_{k=1}^{n} \mu\{I_k\}, \quad n > 1, \text{ integer} \quad (8)$$

for any disjoint right-semi-closed interval sequence $\{I_1, I_2, \cdots, I_n\}$. Then $\mu$ is a finitely additive set function on the algebra $\mathcal{G}(\mathbb{R})$ of all right-semi-closed intervals of $\mathbb{R}$.

**Proof**: (Ash [1]) Comparing to a distribution in

**Definition 2.5**, now the distribution $F$ is defined on the compact space $\mathbb{R}$, thus (1) in **Definition 2.5** is modified to $F(-\infty) = 0$, and $F(+\infty) = 0$ since $F(x)$ is non-decreasing in $x$. Then $\mu\{[-\infty, b]\} = \mu\{(-\infty, b]\} = F(b)$. By the construction, $\mu$ is defined on the algebra (set class) of all right-semi-closed intervals of $\mathbb{R}$ and thus is a set function on algebra $\mathcal{G}(\mathbb{R})$.

**Lemma 2.11**: (Ash [1]) The set function $\mu$ is countably additive on algebra $\mathcal{G}(\mathbb{R})$.

**Proof**: See Ash [1].

**Theorem 2.12**: (Ash [1]) Let $F$ be a right-continuous non-decreasing normed function defined on $\mathbb{R}$, and let $\mu\{(a, b]\} = F(b) - F(a), \forall a, b \in \mathbb{R}$. There is a unique extension of $\mu$ to a Lebesgue-Stieltjes measure on $\mathcal{B}(\mathbb{R})$.

**Proof**: Ash [1] showed this the in following way: let $\mathcal{G}(\mathbb{R})$ be the algebra of all right-semi-closed intervals of $\mathbb{R}$ and extend $\mu$ to the algebra (set class of all right-semi-closed intervals of $\mathbb{R}$), $\mathcal{G}(\mathbb{R})$. By defining the map

$$\begin{align*}
(a, b] &\to (a, b], \text{ if } a, b \in \mathbb{R} \text{ or if } b \in \mathbb{R}, a = -\infty, \\
(a, \infty] &\to (a, \infty], \text{ if } a \in \mathbb{R} \text{ or if } a = -\infty.
\end{align*}$$

we establishes a one-to-one, $\mu$-preserving correspondence between a subset of $\mathcal{G}(\mathbb{R})$ and $\mathcal{G}(\mathbb{R})$. In terms of **Lemma 2.13**, (Ash [1]) $\mu$ is countably additive on
algebra $\mathfrak{A}(\mathbb{R})$. By the Carathéodory extension theorem, $\mu$ has a unique extension to $\mathcal{B}(\mathbb{R})$. Note that

$$\mu\{[a,b]\} = F(b) - F(a), \ \forall a, b \in \mathbb{R}$$

According to Definition 2.9, $\mu$ is a Lebesgue-Stieltjes measure on $\mathcal{B}(\mathbb{R})$.

Remark 2.14: Let $F$ be a distribution, then $F$ is continuous at $x$ if and only if $\mu\{\{x\}\} = 0$, the magnitude of $F$ at $x$ coincides with the measure of $\{x\}$.

Remark 2.15: The functional form of $\mu$ may be

$$\mu\{I\} = \int_I f(s) \, ds$$

where $f$ is an integrable nonnegative function and $I$ is an arbitrary semi-closed interval of $\mathbb{R}$. Eq.(9) helps to trace back the link between $\mu$ and Lebesgue-Stieltjes measure, even Lebesgue (if $f(s) = 1$). (Also see (Ash [1]))

III. Uncertain Measure and Uncertain Variable

Uncertain measure (Liu [6, 7]) is an axiomatically defined set function mapping from a $\sigma$-algebra of a given space (set) to the unit interval $[0,1]$, which provides a measuring grade system of an uncertain phenomenon and facilitates the formal definition of an uncertain variable.

Let $\Xi$ be a nonempty set (space), and $\mathfrak{A}(\Xi)$ the $\sigma$-algebra on $\Xi$. Each element, let us say, $A \subset \Xi$, $A \in \mathfrak{A}(\Xi)$ is called an uncertain event. A number denoted as $\lambda\{A\}$, $0 \leq \lambda\{A\} \leq 1$, is assigned to event $A \in \mathfrak{A}(\Xi)$, which indicates the uncertain measuring grade with which event $A \in \mathfrak{A}(\Xi)$ occurs. The normal set function $\lambda\{A\}$ satisfies following axioms given by Liu [6]:

Axiom L.1: (Normality) $\lambda\{\Xi\} = 1$.

Axiom L.2: (Monotonicity) $\lambda\{\cdot\}$ is non-decreasing, i.e., whenever $A \subset B$, $\lambda\{A\} \leq \lambda\{B\}$.

Axiom L.3: (Self-Duality) $\lambda\{\cdot\}$ is self-dual, i.e., for any $A \in \mathfrak{A}(\Xi)$, $\lambda\{A\} + \lambda\{A^c\} = 1$.

Axiom L.4: ( $\sigma$-Subadditivity) $\lambda\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \lambda\{A_i\}$ for any countable event sequence $\{A_i\}$.

Definition 3.1: (Liu [6]) Any set function $\lambda: \mathfrak{A}(\Xi) \rightarrow [0,1]$ satisfies Axioms L.1-L.4 is called an uncertain measure. The triple $(\Xi, \mathfrak{A}(\Xi), \lambda)$ is called the uncertain measure space.

Definition 3.2: (Liu [6, 7]) An uncertain variable $\xi$ is a measurable mapping, i.e., $\xi: (\Xi, \mathfrak{A}(\Xi)) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, where $\mathcal{B}(\mathbb{R})$ denotes the Borel $\sigma$-algebra on $\mathbb{R} = (-\infty, +\infty)$.

Remark 3.3: The fundamental difference between a random variable and an uncertain variable is the measure space on which they are defined. In the triples, the first two factors are similar in formation: the set and the $\sigma$-algebra on the set. The third factor in the triples: the measures defined on the $\sigma$-algebras are not similar. The former (i.e. the probability measure) obeys $\sigma$-additivity and the later (i.e. the uncertain measure) obeys $\sigma$-subadditivity. The way for specifying measure inevitably has impacts on the behavior of the measurable function on the triple.

Definition 3.4: (Liu [6]) The uncertain distribution $\Psi: \mathbb{R} \rightarrow [0,1]$ of an uncertain variable $\xi$ on $(\Xi, \mathfrak{A}(\Xi), \lambda)$ is

$$\Psi(x) = \lambda\{\tau \in \Xi | \xi(\tau) \leq x\}$$

For the uncertain measure, as an axiomatic measure development, the set class, $\sigma$-algebra $\mathfrak{A}(\Xi)$ plays the critical roles in defining set function - uncertain measure $\lambda$ as well as those in defining the measurability of uncertain variable. The roles are identically the same as to the roles played by a $\sigma$-algebra in probability measure development.

Definition 3.5: (Liu [6]) An $n$-dimensional uncertain vector from an uncertain measure space $(\Xi, \mathfrak{A}(\Xi), \lambda)$ to the set of $n$-dimensional real-valued vector, i.e., for Borel set $B$ of $\mathbb{R}^{+}$, the set

$$\{\xi \in B\} = \{\tau \in \Xi | \xi(\tau) \in B\}$$

is an event.

Theorem 3.6: (Liu [6]) Let $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^{T}$ be an uncertain vector, and $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ a measurable function. Then $f(\xi)$ is an uncertain variable such that

$$\lambda\{f(\xi) \in B\} = \lambda\{\xi \in f^{-1}(B)\}$$
for any Borel set \( B \) of \( \mathbb{R}^n \).

**Proof:** See Liu [6].

### IV. Peng and Iwamura’s Necessary and Sufficient Conditions for Uncertain Distribution

Similar to probability theory, in uncertain theory an uncertain variable is a measurable mapping which is characterized by the membership of the pre-image of event (a Borel set) \( B = (-\infty, r] \) under the uncertain variable \( \xi \) to the \( \sigma \)-algebra \( \mathcal{A}(\Xi) \). In other words,

\[
\forall B \in \mathcal{B}(\mathbb{R}), \{ \tau \in \Xi : \xi \in B \} \in \mathcal{A}(\Xi). \tag{13}
\]

The measurability of uncertain variable \( \xi \) definitely induces a measure on the measurable space \( (\mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R})) \). Let us denote the induced measure \( \nu \). Similar to the probabilistic case in Eq. (5), for \( B \in \mathcal{B}(\mathbb{R}) \) the induced measure is

\[
\nu([a, b]) = \lambda \{ \tau \in \Xi : \xi(\tau) \leq b \} - \lambda \{ \tau \in \Xi : \xi(\tau) \leq a \} \tag{14}
\]

Similarly, the uncertain distribution is defined by the induced measure

\[
\mathcal{Y}(r) = \nu([-\infty, r]) = \lambda \{ \tau \in \Xi : \xi(\tau) \leq r \} \tag{15}
\]

The uncertain distribution defined by the uncertain variable on the uncertain space \( (\Xi, \mathcal{A}(\Xi), \lambda) \) in terms of the induced uncertain measure \( \nu \) on \( \mathcal{B}(\mathbb{R}) \).

Now, let investigate the converse problem: given a normed function, is it possible to define an (induced) uncertain measure so that in terms of measurable mapping, an uncertain variable can be defined?

**Theorem 4.1:** Let \( \mathcal{Y}(\cdot) \) be an non-decreasing normed function with

\[
\mathcal{Y}(-\infty) = 0, \mathcal{Y}(+\infty) = 1. \tag{17}
\]

Then a set function \( \nu \) on the measurable space \( (\mathbb{R}, \mathcal{B}(\mathbb{R})) \) is an uncertain measure defined by the function \( \mathcal{Y}(\cdot) \).

**Proof:** First, we construct an uncertain measure \( \nu \) on the measurable space \( (\mathbb{R}, \mathcal{B}(\mathbb{R})) \) in terms of Peng and Iwamura’s [10] approach.

Define three auxiliary set functions by the function \( \mathcal{Y} \).

The first auxiliary set function \( v_1 : \mathcal{B}(\mathbb{R}) \to [0, 1] \), is defined by

\[
v_1(B) = \begin{cases} 1 - \lim_{x \to \inf B} \mathcal{Y}(x) & \text{if } \inf B \in B \\ 1 - \mathcal{Y}(\inf B) & \text{otherwise} \end{cases} \tag{18}
\]

which defines the value of the set function in terms of the value of the distribution function at the (complement of) the infimum of a Borel set.

The second auxiliary set function \( v_2 : \mathcal{B}(\mathbb{R}) \to [0, 1] \), is defined by

\[
v_2(B) = \mathcal{Y}(\sup B) \tag{19}
\]

which defines the value of the set function in terms of the value of the distribution function at the supremum of a Borel set.

The third auxiliary set function \( v_3 : \mathcal{B}(\mathbb{R}) \to [0, 1] \), is defined by

\[
v_3(B) = \inf_{(a, b) \in B} \{ \mathcal{Y}(a) + 1 - \mathcal{Y}(b) \} \tag{20}
\]

which defines the value of the set function in terms of the infimum value of the distribution function at the endpoints of all the semi-closed interval contained by the complement of the a Borel set.

Then, we define a set function \( v^* \{ \cdot \} \) on \( \mathcal{B}(\mathbb{R}) \):

\[
v^*\{ B \} = v_1\{ B \} \land v_2\{ B \} \land v_3\{ B \}, \forall B \in \mathcal{B}(\mathbb{R}) \tag{21}
\]

Then, for any Borel set \( B \in \mathcal{B}(\mathbb{R}) \), define a set function:

\[
v\{ B \} = \begin{cases} v^*\{ B \} & \text{if } v^*\{ B \} < 0.5 \\ 1 - v^*\{ B^c \} & \text{if } v^*\{ B^c \} < 0.5 \\ 0.5 & \text{otherwise} \end{cases} \tag{22}
\]

Then \( \nu \{ \cdot \} \) on the measurable space \( (\mathbb{R}, \mathcal{B}(\mathbb{R})) \) is the uncertain measure (set function) defined by the increasing normed function \( \mathcal{Y}(\cdot) \).
Now, it is ready to verify that $\nu\{\cdot\}$ satisfies Liu’s [6] Axiom L.1 - Axiom L.4 on space $\mathbb{R}, \mathfrak{B}(\mathbb{R})$.

1. Normality. The space is $\mathbb{R}$ (not the space $\Xi$). For $B = \mathbb{R}$, $\inf \mathbb{R} = -\infty \not\in \mathbb{R}$, then by the normality of $\mathfrak{Y}$

$$v_1(\mathbb{R}) = 1 - \mathfrak{Y}(\inf \mathbb{R})$$

$$= 1 - \mathfrak{Y}(\infty) = 1$$

$$v_2(\mathbb{R}) = \mathfrak{Y}(\sup \mathbb{R})$$

As to $v_1(\mathbb{R})$, the complement of $\mathbb{R}$ is $\emptyset$ (empty set), therefore, interval $[a,b] \subset \emptyset$, which implies $v_1(\mathbb{R}) = 1$.

Accordingly, $v^*\{\mathbb{R}\} = 1$, which implies $v^*(\mathfrak{Y}) = v^*(\emptyset) = 0 < 0.5$

Thus, $v(\mathbb{R}) = 1 - v^*(\mathfrak{Y}) = 1$.

2. Monotonicity. For $\forall A, B \in \mathfrak{B}(\mathbb{R})$, and $A \subset B$, then $\inf A \geq \inf B$ in general. Thus, $\mathfrak{Y}(\inf A) \geq \mathfrak{Y}(\inf B)$, which implies $v_1(A) \leq v_1(B)$. Further, $\mathfrak{Y}(\sup A) \leq \mathfrak{Y}(\sup B)$, for $A \subset B$, which implies $v_2(A) \leq v_2(B)$. The fact $A \subset B$ implies $A' \supset B'$, thus for all intervals contained in $A'$ or $B'$, $[a_x, b_x] \supset [a_y, b_y]$, adding the fact of non-decreasing function $\mathfrak{Y}$, we have

$$\mathfrak{Y}(a_x) \leq \mathfrak{Y}(a_y), \mathfrak{Y}(b_x) \geq \mathfrak{Y}(b_y)$$

then

$$\mathfrak{Y}(a_x) \leq \mathfrak{Y}(a_y), 1 - \mathfrak{Y}(b_x) \leq 1 - \mathfrak{Y}(b_y)$$

And finally

$$\inf \left\{ \mathfrak{Y}(a_x) + 1 - \mathfrak{Y}(b_x) \right\} \leq \inf \left\{ \mathfrak{Y}(a_y) + 1 - \mathfrak{Y}(b_y) \right\}$$

which concludes $v_1(A) \leq v_1(B)$. The above arguments lead the inequality $v^*(A) \leq v^*(B)$ since $v^*$ takes the lower value of $v_1, v_2,$ and $v_3$. Now, we can conclude that $v(A) \leq v(B)$.

3. Self-Duality. This fact can be checked directly from the expression of $v^*$.

$$v(B) = \begin{cases} 
\nu^*\{B\} & \text{if } \nu^*\{B\} < 0.5 \\
1-\nu^*\{B^c\} & \text{if } \nu^*\{B^c\} < 0.5 \\
0.5 & \text{otherwise} 
\end{cases}$$

Notice a symmetry between $B$ and $B^c$, we always have

$$\nu(B) + \nu(B^c) = 1.$$
Case 2. Assume \( \nu(B_i) \geq 0.5 \) and \( \nu(B_i) < 0.5 \) for all \( i > 1 \). For \( \nu(B_i) \geq 0.5 \), \( B_i \subseteq B \), we have \( \nu(B) \geq 0.5 \). Notice that

\[
B_i' \subseteq B' \bigcup \left( \bigcup_{i=1}^\infty B_i \right).
\]

In terms of Case 1 and monotonicity of measure \( \nu \), we have

\[
\nu(B_i') \leq \nu(B') + \sum_{i=1}^\infty \nu(B_i).
\]

which implies

\[
\nu(B) = \nu\left( \bigcup_{i=1}^\infty B_i \right) \leq \sum_{i=1}^\infty \nu(B_i).
\]

Case 3. Assume at least two of \( \nu(B_i) \) are greater than 0.5. Obviously,

\[
\nu(B) \leq 1 \leq \sum_{i=1}^\infty \nu(B_i).
\]

In summary, \( \nu \) satisfies the normality, monotonicity, self-duality, and countable subadditivity axioms, that is, \( \nu \) is actually an uncertain measure on \( \mathcal{B}(\mathbb{R}) \).

**Remark 4.2:** In the proof of Theorem 4.1, nearly all the idea, particularly, the construction of an uncertain measure from a normed non-decreasing function and the \( \sigma \)-subadditivity proof are identical from Peng and Iwamura [10] and also Liu and Ha [9]. However, it is necessary to emphasize that the uncertain measure \( \nu \) defined from a normed non-decreasing function is defined on the Borel \( \sigma \)-algebra not in the “original” uncertain space \( (\Xi, \mathcal{A}(\Xi), \lambda) \). The defined or induced measure space is \( (\mathbb{R}, \mathcal{B}(\mathbb{R}), \nu) \). An uncertain variable \( \xi \) is measurable mapping such that

\[
\lambda \circ \xi^{-1} = \nu
\]

Therefore the “original” uncertain measure can be recovered by

\[
\lambda = \nu \circ \xi
\]

**Definition 4.3:** The function on \( \mathbb{R} \), denoted by \( \gamma \), is an uncertain distribution function if and only \( \gamma \) satisfies following two conditions:

1. \( \lim_{x \to -\infty} \gamma(x) = 0 \), \( \lim_{x \to +\infty} \gamma(x) = 1 \).
2. \( \gamma(x) \) is non-decreasing in \( x \).

**Remark 4.4:** Now we can address the question what is the difference between uncertain distributions and a probability distributions. In terms of Definition 2.5 and Definition 4.3, we can realize that both of them are normed non-decreasing functions but the probability distributions require right-continuity property while the uncertain distributions do not. Any probability distribution can define an uncertain measure on \( \mathcal{B}(\mathbb{R}) \) and consequently an uncertain variable. However, an uncertain distribution is not necessarily able define a probability measure on \( \mathcal{B}(\mathbb{R}) \) and consequently a random variable. The uncertain distributions are less restrictive than these of the probability distributions and inevitably require more conditions for fully specifying them. applied into wider modeling applications.

In probability theory, random variable is classified as discrete random variable, continuous random variable, or mixed one. The continuous random variable can be further classified to be continuous or absolutely continuous. The absolutely continuous random variable has a distribution function \( F \) satisfying

\[
F(x) = \int_{-\infty}^x f(s)ds
\]

where \( f : \mathbb{R} \to \mathbb{R}^+ \), being a continuous function. The probability measure defining an absolutely continuous random variable at any point of real line must be null, i.e., \( P_x \{ \{x\} \} = 0 \), \( \forall x \in \mathbb{R} \).

In uncertain theory, the classification of uncertain variable is parallel to those in random variable. Therefore, we can state a theorem as following:

However, the critical difference in distributions between random variable and uncertain variable is that the former restricts the distribution function to be right-continuous while the later has no restrictions at all. In fact, a probability distribution must be an uncertainty distribution. But an uncertainty distribution is not necessarily a probability distribution. However, probability distribution may produce the values of \( \Pr\{\xi \in B\} \) for any Borel set \( B \in \mathcal{B}(\mathbb{R}) \). But Liu’s [4, 5] definition of an uncertainty distribution may produce only the values \( \lambda\{\xi \leq \alpha\} \) and \( \lambda\{\xi > \alpha\} \), which is not able to fully characterize an uncertain variable, particularly, the uncertain measure underlying the distribution mechanism. Table 1 gives basic comparisons.
TABLE I: Basic comparisons between uncertain variable and random variable

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Uncertain Variable</th>
<th>Random Variable</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi$</td>
<td>$\xi: (\Xi, \mathcal{A}, \lambda) \to (\mathbb{R}, \mathcal{B}, \nu)$</td>
<td>$X: \Omega \to \mathbb{R}$</td>
</tr>
<tr>
<td>Mapping</td>
<td>$\nu = \lambda \circ \xi^{-1}$</td>
<td>$\mu = P \circ X^{-1}$</td>
</tr>
<tr>
<td>Distribution</td>
<td>$Y(x) = \lambda(\xi \leq x)$</td>
<td>$F(x) = P(\xi \leq x)$</td>
</tr>
<tr>
<td>Absolutely continuous</td>
<td>$Y(x) = \int_{-\infty}^x \lambda(s) ds$</td>
<td>$F(x) = \int_{-\infty}^x f(s) ds$</td>
</tr>
<tr>
<td>Singleton</td>
<td>$\lambda({x})$ is not necessarily zero even $Y$ is absolutely continuous</td>
<td>$\Pr({x}) = 0$ if $F$ is absolutely continuous</td>
</tr>
<tr>
<td>Full specification</td>
<td>$\lambda(\xi = x)$, $\lambda(\xi &lt; x)$, $\lambda(\xi \geq x)$</td>
<td>$P(\xi = x)$, $P(\xi &lt; x)$</td>
</tr>
</tbody>
</table>

In fact, a probability distribution must be an uncertainty distribution. But an uncertainty distribution is not necessarily a probability distribution, since $\lambda(\{x\})$ is not necessarily zero. We give a definition which fully specifies the uncertain distribution with the underlying uncertain measure.

**Definition 4.5**: The uncertain distribution $Y: \mathbb{R} \to [0, 1]$ of an uncertain variable $\xi$ on $\Omega$ is a non-decreasing function such that for all $x \in \mathbb{R}$,

$$ Y(x) = \lambda(\xi \leq x), $$

and

$$ Y(\{x\}) = \lambda(\xi = x), \quad Y(\{x\}) = \lambda(\xi > x). $$

**Remark 4.6**: After reviewing Peng and Iwamura’s [10] sufficient and necessary conditions, we should fully aware to specify an uncertain distribution is more complicated than that of a probability distribution, the underlying uncertain measures $\lambda(\xi = x), \lambda(\xi < x), \lambda(\xi > x)$ at any point $x$ must be specified. This gives a tough and delicate challenge to uncertainty statistics, in which estimating an uncertain distribution is an essential subject.

V. The Identification Functions

Section IV reveals the general features of uncertain distributions, and also the construction of uncertain measure on Borel $\sigma$-algebra from the distribution. Nevertheless, the general discussions left some undressed problems, for example, are there any function subclasses with explicit forms to identify or to define uncertain variables and uncertain measures? The answer is yes, although partially. Liu [6] pointed out three types of identification functions can characterize uncertain variables. Let us examine the characteristics and the roles of identification functions.

**Definition 5.1**: (Identification function of the first kind) If function $\iota: \mathbb{R} \to \mathbb{R}^+$ satisfying

$$ \sup_{x,y} \{\iota(x) + \iota(y)\} = 1 $$

Then, $\iota(\cdot)$ is termed as the identification function of the first kind for an uncertain variable $\xi$.

**Theorem 5.2**: If $\iota(\cdot)$ is an identification function of the first kind, then for all $B \subseteq \mathbb{R}$, an uncertain measure is defined by

$$ \lambda(B) = \left\{ \begin{array}{ll} \sup_{x \in B} \iota(x) & \text{if } \sup_{x \in B} \iota(x) < 0.5 \\ 1 - \sup_{x \in B} \iota(x) & \text{if } \sup_{x \in B} \iota(x) \geq 0.5 \end{array} \right. $$

**Proof**: The set function $\lambda(\cdot)$ satisfies normality, monotonicity, self-duality and $\sigma-$sub-additivity with the support of equality: $\sup_{x \neq y} \{\iota(x) + \iota(y)\} = 1$, thus it is an uncertain measure. An uncertain variable $\xi$ mapping from the uncertain space $(\Xi, \mathcal{A}(\Xi), \lambda)$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \nu)$.

**Remark 5.3**: Liu [6] lists three examples of uncertain variable with the identification function of the first kind: rectangular (identification function),

$$ \iota(x) = 0.5 \delta_{[x]}(x), \quad x \in \mathbb{R} $$

which satisfies Eq. (26) defines the linear uncertain variable with uncertain distribution:

$$ \Phi(\xi) = \lambda(\xi \leq x) = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b, \\ 1 & \text{if } x > b \end{cases} $$

triangular (identification function),
which defines the four piece-wise linear uncertain distribution

$$
\Phi(x) = \lambda\{\xi \leq x\} = \begin{cases} 
0 & \text{if } x \leq a \\
\frac{x-a}{2(b-a)} & \text{if } a < x \leq b \\
\frac{x-c-2b}{2(c-b)} & \text{if } b < x \leq c \\
1 & \text{if } x > c 
\end{cases}
$$

and trapezoidal (identification function)

$$
\iota(x) = \frac{x-a}{2(b-a)} \eta_{[a,b]}(x) + \frac{c-x}{2(c-b)} \eta_{[c,d]}(x), \quad x \in \mathbb{R}
$$

which defines the five piece-wise linear uncertain distribution

$$
\Phi(x) = \lambda\{\xi \leq x\} = \begin{cases} 
0 & \text{if } x \leq a \\
\frac{x-a}{2(b-a)} & \text{if } a < x \leq b \\
\frac{x-c-2b}{2(c-b)} & \text{if } b < x \leq c \\
0.5 & \text{if } c < x \leq d \\
1 & \text{if } x > d 
\end{cases}
$$

Remark 5.4: The three identifications in the form are the same as fuzzy membership functions except a scale factor 0.5. Therefore, fuzzy credibility distributions can also be used to define uncertain variables. These uncertain variables are very similar to the corresponding credibilistic fuzzy variables except the credibility measure is defined on power set (which is the largest \(\sigma\)-algebra) while the uncertain measure is defined on a \(\sigma\)-algebra.

Remark 5.5: Examining the way defining the three piecewise linear uncertain distributions, they are not right-continuous functions, and thus they do not define random variables but uncertain variables although they may be redefined or modified in order to be probability distributions.

Definition 5.6: (Identification function of the second kind) If an integrable function \(\rho : \mathbb{R} \rightarrow \mathbb{R}^+\) satisfying

$$
\int_{\mathbb{R}} \rho(s) ds \geq 1
$$

then, \(\iota(\cdot)\) is termed as the identification function of the second kind for an uncertain variable \(\xi\).

**Theorem 5.7:** If \(\rho(\cdot)\) is an identification function of the second kind, then for \(\forall B \subset \mathbb{R}\), an uncertain measure is defined by

$$
\lambda\{B\} = \begin{cases} 
\int_s \rho(s) ds & \text{if } \int_s \rho(s) ds < 0.5 \\
1-\int_s \rho(s) ds & \text{if } \int_s \rho(s) ds < 0.5 \\
0.5 & \text{otherwise} 
\end{cases}
$$

**Proof:** See Liu [6]. The set function \(\lambda\{\cdot\}\) satisfies normality, monotonicity, self-duality and \(\sigma\)-sub-additivity with the support of inequality in Eq. (29), thus it is an uncertain measure. An uncertain variable \(\xi\) mapping from the uncertain space \((\Xi, \mathbb{A}(\Xi), \lambda)\) to \((\mathbb{R}, \mathcal{B}(\mathbb{R}), \nu)\).

Remark 5.8: \(\rho(\cdot)\) may be a simple function, e.g.,

$$
\rho(x) = \sum_{i=1}^{m} x_i \eta_{[a_i, b_i]}(x), \quad x \in \mathbb{R},
$$

or an elementary function, e.g.,

$$
\rho(x) = \sum_{i=1}^{m} x_i \eta_{[a_i, b_i]}(x), \quad x \in \mathbb{R},
$$

It is obvious that if an identification function takes a simple function form or an elementary function form, then a discrete uncertain distribution is identified.

Remark 5.9: Let

$$
\rho(x) = \frac{\pi}{\sqrt{3\sigma}} \frac{1}{\cosh \left( \frac{\pi}{\sqrt{3\sigma}} (x-c) \right)}, \quad x \in \mathbb{R}
$$

then
\[
\lambda(x) = \frac{1}{1 + e^{-\lambda x}}, \quad x \in \mathbb{R}
\]
which defines the normal uncertain distribution

\[
\mu(x) = \lambda_\mu \{ x \leq x \} = \frac{1}{2} \left( 1 + e^{-\sqrt{2\pi} (x - \mu)} \right)
\]
and thus defines the normal uncertain variable.

**Theorem 5.10**: If the identification \( \rho \) is continuous with equality \( \int \rho = 1 \). Then the uncertain distribution being identified by \( \rho \) is also absolutely continuous probability distribution.

**Definition 5.11**: (Identification function of the third kind)
If a function pair \( (\tau, \rho) \) with a nonnegative function \( \tau : \mathbb{R} \to \mathbb{R}^+ \) and an integrable function \( \rho : \mathbb{R} \to \mathbb{R}^+ \) satisfying

\[
\begin{align*}
\sup_{x \in B} \tau(x) + \int_B \rho(s) ds &\geq 0.5 \\
\text{and/or} \\
\sup_{x \in \mathbb{R}} \tau(x) + \int_{\mathbb{R}} \rho(s) ds &\geq 0.5
\end{align*}
\]

for \( B \in \mathcal{B}(\mathbb{R}) \). Then, the function pair \( (\tau, \rho) \) is termed as the identification function of the third kind for an uncertain variable \( \xi \).

**Theorem 5.12**: If \( (\tau, \rho) \) is an identification function of the third kind, then for \( \forall B \subset \mathbb{R} \), an uncertain measure is defined by

\[
\lambda_\nu(B) = \begin{cases} 
\sup_{x \in B} \tau(x) + \int_B \rho(s) ds & \text{if } \sup_{x \in B} \tau(x) + \int_B \rho(s) ds < 0.5 \\
1 - \sup_{x \in B} \tau(x) - \int_B \rho(s) ds & \text{if } \sup_{x \in B} \tau(x) + \int_B \rho(s) ds < 0.5 \\
0.5 & \text{otherwise}
\end{cases}
\]

VI. The Essential Form of an Uncertain Distribution

Because the criticality of the uncertain distribution be played in statistical estimation of uncertain linear model theory, it is necessary to further investigate the uncertain measure on which an uncertain variable and its distribution is defined.

**Definition 4.7** is implicitly characterizes point wise an uncertain distribution and its specific link to the uncertain measure which generates the distribution function.

Let us examine a discrete uncertain distribution - uncertain measure example.
Example 6.1: The set $\Xi = \{ \theta_1, \theta_2, \theta_3, \theta_4 \}$, the $\sigma$-algebra on set $\Xi, A$, contains $2^4 = 16$ elements. They are listed as following table:

<table>
<thead>
<tr>
<th>$\theta_i$</th>
<th>Form of subsets</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 $\theta_1$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>1 $\theta_1$</td>
<td>${ \theta_1, \theta_2, \theta_3, \theta_4 }$</td>
</tr>
<tr>
<td>2 $\theta_1$</td>
<td>${ \theta_1, \theta_2, \theta_3, \theta_4 }, { \theta_1, \theta_2, \theta_4 }, { \theta_1, \theta_2, \theta_3 }$</td>
</tr>
<tr>
<td>3 $\theta_1$</td>
<td>${ \theta_1, \theta_2, \theta_3, \theta_4 }, { \theta_1, \theta_2, \theta_4 }$</td>
</tr>
<tr>
<td>4 $\theta_1$</td>
<td>$\Xi = { \theta_1, \theta_2, \theta_3, \theta_4 }$</td>
</tr>
</tbody>
</table>

The uncertain variable, denoted by $\xi$, which takes values on $X = \{0, 1, 2, 3, 4\}$, let us define the uncertain distribution $\Psi$:

| $\xi$ | $\Psi(\xi = \lambda | \xi < x)$ | $\Psi(\xi = \lambda | \xi = x)$ | $\Psi(\xi = \lambda | \xi > x)$ |
|-------|-----------------|-----------------|-----------------|
| 0     | 0.00            | 0.00            | 1.00            |
| 1     | 0.20            | 0.05            | 0.70            |
| 2     | 0.45            | 0.05            | 0.55            |
| 3     | 0.70            | 0.05            | 0.20            |
| 4     | 0.95            | 0.05            | 0.00            |

It therefore specifies the uncertain distribution:

$$
\Psi(0) = \lambda | \xi = 0 = 0, \\
\Psi(1-) = 0.20, \quad \Psi(1) = 0.25, \quad \Psi(1+) = 0.30, \\
\Psi(2-) = 0.45, \quad \Psi(2) = 0.50, \quad \Psi(2+) = 0.55, \\
\Psi(3-) = 0.70, \quad \Psi(3) = 0.75, \quad \Psi(3+) = 0.80, \\
\Psi(4-) = 0.95, \quad \Psi(4) = 1.00
$$

Remark 6.2: It is easily seen that the uncertain distribution $\Psi$ defined in Example 6.1 satisfies Definition 4.3 and Definition 4.5. It can further be verified that the uncertain distribution $\Psi$ is neither left continuous nor right-continuous. In some sense, the general form of an uncertain distribution should look and behave like $\Psi$.

Definition 6.3: (Essential Form of an Uncertain Variable) Let $\xi$ be an uncertain variable with essential form, which takes values from ascending ordered domain set $D = \{ c_0, c_1, \cdots, c_n \}$ with the uncertain distribution $\Psi$ defined by

$$
\Psi(c_i) = \lambda \{ \xi \leq c_i \} = 0, \\
\Psi(c_i^-) = \nu_i, \quad \Psi(c_i) = \psi_i, \quad \Psi(c_i^+) = \psi_i, \\
\Psi(c_i^-) = \nu_i, \quad \Psi(c_i) = \psi_i, \quad \Psi(c_i^+) = \psi_i
$$

such that $\psi_i < \psi_j < \psi_j, \quad i = 1, 2, \cdots, n$. Furthermore, it requires $\pi_i = \lambda \{ \xi = c_i \} \in (0, \nu_i, -\psi_i), \quad i = 1, 2, \cdots, n$.

Theorem 6.4: Let $\xi$ be an uncertain variable with essential form, which takes values from ascending ordered domain set $D = \{ c_0, c_1, \cdots, c_n \}$ with the uncertain distribution $\Psi$ satisfying the following necessary and sufficient conditions:

(1) $\Psi(c_i) = \lambda \{ \xi \leq c_i \} = 0$

(2) For $i = 1, 2, \cdots, n - 1$, $\Psi(c_i^-) = \lambda \{ \xi < c_i \} = \nu_i$, $\Psi(c_i) = \lambda \{ \xi = c_i \} = \pi_i$, $\Psi(c_i^+) = \lambda \{ \xi > c_i \} = \psi_i$

(3) $\Psi(c_n^-) = \lambda \{ \xi < c_n \} = \nu_n$, $\Psi(c_n) = \lambda \{ \xi = c_n \} = \pi_n$, $\Psi(c_n^+) = \lambda \{ \xi \leq c_n \} = 1.00$

(4) The uncertain measure of singleton $\{ c_i \}$

$$
\pi_i = \lambda \{ \xi = c_i \} \in (0, \nu_i, -\psi_i), \quad i = 1, 2, \cdots, n
$$

must be specified.

Theorem 6.5: The expectation of an uncertain distribution $\Psi$, denoted as $E_\Psi[\xi]$, is given by

$$
E_\Psi[\xi] = \sum_{i=0}^{n} w_i c_i
$$

where

$$
w_i = \max_{c_{j \leq c_i}} \{ \pi_j \mid c_j \leq c_i \} \land 0.5 - \max_{c_{j > c_i}} \{ \pi_j \mid c_j > c_i \} \land 0.5 \\
+ \max_{c_{j \leq c_i}} \{ \pi_j \mid c_j \geq c_i \} \land 0.5 - \max_{c_{j > c_i}} \{ \pi_j \mid c_j \geq c_i \} \land 0.5
$$

$i = 1, 2, \cdots, n$.
Remark 6.6: The proof of Theorem 6.4 is just applying Liu’s [6] definition of uncertain expectation to discrete uncertain variable with neither left-continuity nor right-continuity:

$$E[\xi] = \int_0^\infty \lambda_\xi \{ \xi \geq s \} ds - \int_{-\infty}^0 \lambda_\xi \{ \xi \leq s \} ds . \quad (53)$$

Theorem 6.7: Let $\rho$ be a continuous identification function of the second kind, i.e., $\int_\mathbb{R} \rho(s) ds \geq 1$, then the singleton of real line have uncertain measure zero, i.e.

$$\lambda_\left\{ \{x\} \right\} = 0, \forall x \in \mathbb{R} \quad (54)$$

Proof: For any $\varepsilon > 0$, let us investigate the uncertain measure of small interval $[x-\varepsilon/2, x+\varepsilon/2]$. In terms of Liu’s [6] Second Measure Reversion Theorem: Let $\xi$ be an uncertain variable with the identification of the second kind, $\rho$. Then for any Borel set $B \in \mathcal{B}$,

$$\lambda_\xi \{ x \in B \} = \begin{cases} \int_B \rho(x) dx & \text{if } \int_B \rho(x) dx < 0.5 \\ 1 - \int_B \rho(x) dx & \text{if } \int_B \rho(x) dx < 0.5 \\ 0.5 & \text{otherwise} \end{cases} \quad (55)$$

Here let $B = [x-\varepsilon/2, x+\varepsilon/2]$, then integral $\int_B \rho(x) dx$ can be expressed by

$$\int_{x-\varepsilon}^{x+\varepsilon} \rho(x) dx = \rho(x') \varepsilon, x' \in [x-\varepsilon/2, x+\varepsilon/2] \quad (56)$$

This is an application of the mean value theorem in mathematics. The fact that $\rho(x)$ is continuous secures the validity.

Let $\varepsilon \to 0$, therefore

$$\lambda_\xi \{ x \in [x-\varepsilon/2, x+\varepsilon/2] \} \to 0 \quad (57)$$

which implies

$$\lambda_\left\{ \{x\} \right\} = 0, \forall x \in \mathbb{R} \quad (58)$$

Corollary 6.8: Let the standard normal density function be

$$\lambda(x) = \frac{\pi}{\sqrt{3}} \frac{1}{1 + \text{ch} \left( \frac{\pi}{\sqrt{3} \sigma} \right)}, x \in \mathbb{R}$$

then the singleton of real line have uncertain measure zero, i.e. $\lambda_\left\{ \{x\} \right\}$.

$$\lambda_\left\{ \{x\} \right\} = 0, \forall x \in \mathbb{R} \quad (59)$$

Proof: $\lambda(x)$ is continuous and is an identification of the second kind since $\int_\mathbb{R} \lambda(x) dx = 1$. Thus applying Theorem 6.6, we reach the conclusion.

Theorem 6.9: If an uncertain distribution $\Upsilon$ can be expressed by a Riemann integral

$$\Upsilon(x) = \int_{-\infty}^x \zeta(s) ds$$

where $\zeta : \mathbb{R} \to \mathbb{R}^+$, $\zeta \in C^0(\mathbb{R})$ (which is a class of continuous functions). Then

$$\lambda_\left\{ a \leq x \leq b \right\} = \lambda_\left\{ -\infty < x \leq b \right\} - \lambda_\left\{ -\infty < x \leq a \right\} = \int_a^b \zeta(s) ds \quad (60)$$

Furthermore, $\lambda_\left\{ \{x\} \right\} = 0, \forall x \in \mathbb{R}$.

VII. Concluding Remarks

In this paper, based on reviewing and comparing Kolmogorov’s [3] axiomatic probability measure and Liu’s [6] axiomatic uncertain measure theoretic foundation, we carry on a comparative study between the probability distribution and uncertain distribution. By the comparisons, the critical finding is the essential form of the general uncertain distribution is identified, which is neither left continuous nor right-continuous. We expect this paper helps to clarify the basic concept of uncertain distribution theory and start from here to explore the statistical estimation on uncertain distributions and general uncertainty decision theory.

Acknowledgements

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References


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